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On cyclic colorings and their generalizations

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Abstract

A cyclic coloring is a vertex coloring such that vertices in a face receive different colors. Let Δ be the maximum face degree of a graph. This article shows that plane graphs have cyclic $\frac{9}{5}\Delta$-colorings, improving results of Ore and Plummer, and of Borodin. The result is mainly a corollary of a best-possible upper bound on the minimum cyclic degree of a vertex of a plane graph in terms of its maximum face degree. The proof also yields results on the projective plane, as well as for d-diagonal colorings. Also, it is shown that plane graphs with $\Delta = 5$ have cyclic 8-colorings. This result and also the $\frac{9}{5}\Delta$ result are not necessarily best possible.

Keywords: Cyclic coloring; Plane graphs; Degree of vertices

1. Introduction

This article will consider connected, non-empty graphs embedded on surfaces, which may have loops or multiple edges. The *degree* of a vertex x, in symbols $\deg(x)$, will be the number of edges incident with x plus the number of loops incident with x. Let an edge be a *coloop* if it is incident with only one face. Then the *degree* of a face A, in symbols $\deg(A)$, will be the number of edges incident with A plus the number of coloops incident with A. Thus, the degrees of vertices and faces are dual concepts. An alternate definition of the degree of face can be made in terms of its *facial walk*, which is the closed walk traversing the boundary of the face. The degree of a face is equal to the length of its facial walk.

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A *cyclic* coloring of an embedded graph is a coloring of the vertices of the graph, such that if two vertices are incident with a common face, they receive different colors. Let a *k-coloring* of a graph G be a coloring of the vertices of G using at most k colors. Given a set T of embedded graphs, the *cyclic chromatic number* of T is the minimum number k, such that every graph in T has a cyclic k-coloring.

Given an embedded graph G, let $\Delta(G)$ (or Δ if the graph is clear from context) be the maximum face degree of G. Ore and Plummer [12] showed that plane graphs have cyclic chromatic number at most 2Δ. A lower bound for this class of graphs is $\lfloor \frac{3}{2}\Delta \rfloor$, which is conjectured to be best possible, (see e.g. [11], p. 37). Borodin [4] improved the upper bound to $2\Delta-3$ for $\Delta \geqslant 8$. Results for small Δ are known. Appel and Haken [1] (see also [14]) proved the four color theorem, which is equivalent to the case $\Delta=3$. Borodin [2] (see also [6]) showed the best possible result for $\Delta=4$, that this class of graphs has cyclic chromatic number 6. He [4] showed that the cyclic chromatic number of plane graphs with $\Delta = 5, 6$, and 7, is at most 9, 11, and 12, respectively. There is one result on the projective plane. Schumacher [18] showed that the cyclic chromatic number of projective plane graphs with $\Delta=4$ is at most 7.

This article proves that plane graphs have cyclic chromatic number at most $\frac{9}{5}\Delta$. The results of Appel and Haken and Borodin above imply this bound for $\Delta \leqslant 5$ and for $7 \leqslant \Delta \leqslant 15$. Thus two cases remain.

The first case considered is for $\Delta \geqslant 16$. Section 3 proves the $\frac{9}{5}\Delta$ result for $\Delta \geqslant 16$, by means of a structural result. Let two vertices x, y of a graph G be *cyclically adjacent* if they are incident with a common face of G. Let the *cyclic degree* of a vertex x (in symbols $\mathrm{cd}(x)$) be the number of vertices which are cyclically adjacent to x. This article defines a function d of the maximum face degree Δ, with the following properties: Every plane or projective plane graph with maximum face degree Δ has a vertex of cyclic degree at most $d(\Delta)$. For every Δ, there is a plane graph and a projective plane graph each of which has maximum face degree Δ and minimum cyclic degree $d(\Delta)$.

The second case considered is for $\Delta=6$. Section 6 proves the $\frac{9}{5}\Delta$ result for $\Delta=6$, that these graphs have cyclic 10-colorings.

As mentioned above, the only cases where the cyclic chromatic number is known precisely are for $\Delta=3$ (because this is equivalent to proper colorings), and for $\Delta=4$ on the plane (as shown by Borodin in [2,6]). A corollary of the structural result of Section 3 is that the cyclic chromatic number of projective plane graphs with $\Delta=5$ is equal to 10. Section 5 gives an improvement on the cyclic chromatic number of plane graphs with $\Delta=5$, that they have cyclic 8-colorings; the best-known lower bound for this case is 7 [13].

Each of these proofs uses the discharging method, the method that was used to prove the four color theorem [1] or [14]. Section 4 shows how the results for $\Delta \geqslant 16$ also apply to the generalization of cyclic colorings known as d-diagonal colorings.

Note that improvements on the cyclic chromatic number have been shown for 3-connected plane graphs by Plummer and Toft [13] and Borodin [7].

2. The discharging method

This section will describe the discharging method, which will be used to obtain the results of this article.

Given an embedded graph G, let the sets of its vertices, edges, and faces be, respectively, $V(G), E(G)$, and $F(G)$ (or V, E, and F, if the graph is clear from the context). Also, let the cardinalities of these respective sets be $v(G), e(G)$, and $f(G)$ (or v, e, and f).

Vertices and faces will frequently be referred to by bounds on their degrees. Thus the following definitions are useful. Let a *k-vertex* be a vertex of degree k. Let an *at most k-vertex*, or an *$\leqslant k$-vertex* for brevity, be a vertex of degree at most k. Similarly, let an *$\geqslant k$-vertex* be a vertex of degree at least k. Let *k-face*, *$\leqslant k$-face*, and *$\geqslant k$-face* be defined similarly.

For a positive real number k, let an embedded graph G be *k-charged* if a function ch_k is defined on $V \cup F$ as follows: For each vertex x of G, let $\mathrm{ch}_k(x) := k - \deg(x)$. For each face A of G, let $\mathrm{ch}_k(A) := k - [(k-2)/2]\deg(A)$. If an embedded graph is k-charged, the function ch_k will be described as the *initial charges* of its vertices and faces. The following lemma is the basic premise of the discharging method.

Lemma 2.1. *Let k be a positive real number, and let G be a k-charged* (*connected*) *graph which is embedded in the plane or the projective plane. Then*

$$\sum_{x \in V \cup F} \mathrm{ch}_k(x) > 0.$$

Proof. Let k and G be as in the statement. If G is plane, then Euler's formula states that $v - e + f = 2$. If G is projective plane (without loss of generality, each face is an open 2-cell), then Euler's formula states that $v - e + f = 1$. In either case, $v - e + f > 0$. Since $k > 0$, this gives $kv - ke + kf > 0$, or $kv - 2e + kf - (k-2)e > 0$. Since $\sum_{x \in V} \deg(x) = 2e$, and $\sum_{A \in F} \deg(A) = 2e$, it follows that $\sum_{x \in V}(k - \deg(x)) + \sum_{A \in F}(k - [(k-2)/2]\deg(A)) > 0$, which is equivalent to the conclusion of the lemma. $\square$

This article will be concerned with 4-charged graphs, where $\mathrm{ch}_4(x) = 4 - \deg(x)$ for vertices and faces, and 6-charged graphs, where $\mathrm{ch}_6(x) = 6 - \deg(x)$ for vertices, and $\mathrm{ch}_6(A) = 6 - 2\deg(A)$ for faces. In these two cases, Lemma 2.1 yields two classic results about plane and projective plane graphs. Since $\sum_{x \in V \cup F} \mathrm{ch}_4(x) > 0$, each such graph must have either an $\leqslant 3$-vertex or an $\leqslant 3$-face. Also, since $\sum_{x \in V \cup F} \mathrm{ch}_6(x) > 0$, each such graph must have either an $\leqslant 5$-vertex or an $\leqslant 2$-face, the latter excluded if the graph is simple.

This article will be concerned with showing the existence of more complicated structures (such as a vertex of low cyclic degree) in these graphs. A method that accomplishes this is known as the discharging method, and will be used in this article. Lemma 2.1 guarantees the existence of one of a number of elements (vertices or faces) which have positive initial charge. A set of discharging rules is defined, which, in certain situations determined by the local structure of the graph, moves the positive charge away from these elements into nearby elements which usually have negative initial charge. As the sum of these modified charges is equal to the sum of the initial charges, there will be an element which has positive modified charge. If the discharging rules are chosen carefully, an examination of each vertex and each face according to its degree will show that either this element has non-positive modified charge, or a desired structure is present in the graph nearby this element.

Each of Sections 3, 5, and 6 contains a list of desired structures and a list of discharging rules. Also, each contains a proof by means of the discharging method which states that after the charge is modified according to the given rules, each element with positive charge guarantees the existence of one of the given configurations.

3. A best-possible upper bound on the minimum cyclic degree

This section gives a best-possible upper bound on the minimum cyclic degree of plane and projective plane graphs in terms of their maximum face degree. Let Σ_0 be the plane, and let $\tilde{\Sigma}_1$ be the projective plane. Given a surface Σ and an integer n, let $D(\Sigma, n)$ be the set of graphs embedded on Σ with maximum face degree n, and then let $\mathrm{cd}(\Sigma,n)$ be $\max_{G\in D(\Sigma,n)}\{\min_{x\in V(G)}\{\mathrm{cd}(x)\}\}$, or ∞ if this is undefined. This section will determine the correct values of $\mathrm{cd}(\Sigma_0,n)$ and $\mathrm{cd}(\tilde{\Sigma}_1,n)$ for all n by means of the discharging method. Note that the proofs of some of the cyclic coloring results mentioned in the introduction imply upper bounds on $\mathrm{cd}(\Sigma_0,n)$ for certain values of n.

Now the discharging rules for this proof will be defined: Note that a vertex can be incident with the same face more than once. In other words, although a j-vertex is surrounded by j faces, these faces are not necessarily distinct. Similarly, k vertices appear in the facial walk of a k-face, but they are not necessarily distinct. Let a 4-charged graph G be *discharged by the first set of rules* if a function ch_4' is defined by modifying ch_4 according to Rules 1–7 below.

Before the rules are stated, an explanation of some of the notation will be given. For example, in Rule 1, x_2 is said to be incident with two $\geqslant 16$-faces. It may be that these two faces are identical (i.e. that x_2 is incident twice with the same face). In this case the rule should be interpreted to mean that x_2 then sends 1 into it twice, and the face receives 1 from each copy of x_2 in its facial walk. The same logic applies to the remaining rules.

In Rules 6 and 7, it is desirable to send charge from one face to another. For instance, in Rule 6, it is desirable to send $\frac{1}{6}$ from A_3 to A_5. On the other hand, it is convenient for the proof to have vertices receive charge only from faces, and vice versa. Thus, the charge of $\frac{1}{6}$ is sent through the vertex x. When counting charge, A_3

will send its charge into x, and A_5 will receive its charge from x. Doing it this way, for instance, allows a 3-face to send $\frac{1}{3}$ to each of its incident vertices of large cyclic degree, as will be discussed in detail in the proof of the theorem below.

1. If a 2-vertex x_2 is incident with two $\geqslant 16$-faces, then let x_2 send 1 into each of them.
2. If a 3-vertex x_3 is incident with two $\geqslant 8$-faces and one $\leqslant 4$-face, then let x_3 send $\frac{1}{2}$ into each $\geqslant 8$-face.
3. If a 3-vertex x_3 is incident with two $\geqslant 7$-faces and one 5-face, then let x_3 send $\frac{2}{5}$ into each $\geqslant 7$-face and $\frac{1}{5}$ into the 5-face.
4. If a 3-vertex x_3 is incident with three $\geqslant 6$-faces, then let x_3 send $\frac{1}{3}$ into each of them.
5. For each 3-face A_3, and for each $\geqslant 5$-vertex x incident with A_3, let A_3 send $\frac{1}{3}$ into x.
6. For each edge α incident with both a 3-face A_3 and an $\geqslant 5$-face A_5, and for each $\leqslant 4$-vertex x incident with α, let A_3 send $\frac{1}{6}$ through x into A_5.
7. For each 4-vertex x incident with a 3-face A_3, an $\leqslant 4$-face A_4, and two $\geqslant 5$-faces A_1, A_2, such that the cyclic order of the faces around x is (A_1, A_2, A_3, A_4), let A_3 send $\frac{1}{6}$ through x into A_1.

Thus some positive charge is sent into vertices and faces which had negative initial charge.

Let the function d be defined as follows: $d(1) = 0$, $d(2) = 1$, $d(3) = 5$, $d(4) = 7$, $d(5) = 9$, $d(6) = 11$, $d(7) = 12$, and if $n \geqslant 8$, then $d(n) = \lfloor \frac{9}{5} n \rfloor - 1$.

Theorem 3.1. *For every positive integer* $n, \mathrm{cd}(\Sigma_0, n) = \mathrm{cd}(\tilde{\Sigma}_1, n) = d(n)$.

Proof. Let a positive integer n be given. The result is trivial if $n \leqslant 2$, thus assume $n \geqslant 3$.

Let $G \in D(\Sigma_0, n) \cup D(\tilde{\Sigma}_1, n)$ be given. Assume that every vertex x of G has $\mathrm{cd}(x) > d(n)$.

If G has an $\leqslant 2$-face, let α be an edge incident with it. Let $H := G - \alpha$. Since $\Delta(H) \leqslant \Delta(G)$, it follows that $d(\Delta(H)) \leqslant d(\Delta(G))$. Also, two vertices are cyclically adjacent in G if and only if they are cyclically adjacent in H. Thus, it suffices to consider graphs without $\leqslant 2$-faces.

Let G be 4-charged, and then discharged by the first set of rules. First, the vertices of G will be examined.

Let x_1 be a 1-vertex of G. Clearly $\mathrm{cd}(x_1) \leqslant d(n)$; this contradicts the assumption that every vertex of G has cyclic degree greater than $d(n)$. Thus G has no 1-vertices.

Let x_2 be a 2-vertex of G. Note $\mathrm{ch}_4(x_2) = 2$. If x_2 is incident with an $\leqslant 15$-face, then $\mathrm{cd}(x_2) \leqslant \min\{n + 11, 2n - 4\} \leqslant d(n)$, contrary to assumption. Thus x_2 is incident with two $\geqslant 16$-faces, it sends out 2 by Rule 1, and $\mathrm{ch}_4'(x_2) = 0$.

Let x_3 be a 3-vertex of G. Note $\mathrm{ch}_4(x_3) = 1$. If x_3 is incident with an $\leqslant 4$-face A_4 and an $\leqslant 7$-face A_7, or if x_3 is incident with a 5-face A_5 and an $\leqslant 6$-face A_6, then (as in the previous paragraph) $\mathrm{cd}(x_3) \leqslant d(n)$, a contradiction. It follows that x_3 sends out 1 by one of Rules 2, 3, or 4, and $\mathrm{ch}'_4(x_3) \leqslant 0$.

Let x_4 be a 4-vertex of G. Here $\mathrm{ch}_4(x_4) = 0$, and since the rules do not affect it, $\mathrm{ch}'_4(x_4) = 0$ as well.

Let x_5 be a 5-vertex of G. Note $\mathrm{ch}_4(x_5) = -1$. If x_5 is incident with at least four 3-faces, then $\mathrm{cd}(x_5) \leqslant d(n)$ as before. Otherwise, x_5 is incident with at most three 3-faces, and since each sends $\frac{1}{3}$ into x_5 by Rule 5, $\mathrm{ch}'_4(x_5) \leqslant 0$.

Let x_6 be a k-vertex of G, where $k \geqslant 6$. Here $\mathrm{ch}_4(x_6) = 4 - k$. At most $\frac{k}{3}$ comes into x_6 by Rule 5, and as no other rule applies, $\mathrm{ch}'_4(x_6) \leqslant 0$.

Thus every vertex of G has non-positive modified charge.

Also, the previous argument that a 2-vertex which is adjacent to an $\leqslant 15$-face has low cyclic degree will be used again without further justification. This will also be true for a 3-vertex adjacent to two faces, the sum of whose degrees is at most 11, and for a 4-vertex adjacent to three faces, the sum of whose degrees is at most 11.

Let A_3 be a 3-face of G. Note $\mathrm{ch}_4(A_3) = 1$. For each vertex x incident with A_3, note that if $\deg(x) \geqslant 5$, then A_3 sends $\frac{1}{3}$ into x by Rule 5. If $\deg(x) = 2$, then $\mathrm{cd}(x) \leqslant d(n)$ as before. If $\deg(x) = 3$, then either $\mathrm{cd}(x) \leqslant d(n)$, or A_3 sends $\frac{1}{3}$ into x by Rule 6. If $\deg(x) = 4$, then either $\mathrm{cd}(x) \leqslant d(n)$, or A_3 either sends $\frac{1}{3}$ into x by Rule 6, or sends $\frac{1}{6}$ into x by Rule 6 and 16 into x by Rule 7. Thus A_3 sends out 1, receives none, and hence $\mathrm{ch}'_4(A_3) = 0$.

Let A_4 be a 4-face of G. Here $\mathrm{ch}_4(A_4) = 0$, and since the rules do not affect it, $\mathrm{ch}'_4(A_4) = 0$ as well.

Let A_5 be a 5-face of G. Note $\mathrm{ch}_4(A_5) = -1$. If there is a 4-vertex x incident with A_5 and two 3-faces, or a 3-vertex x incident with A_5 and a 3-face, then $\mathrm{cd}(x) \leqslant d(n)$. Otherwise, each of the five copies of vertices (which are not necessarily distinct) in the facial walk of A_5 sends at most $\frac{1}{5}$ into A_5, and hence $\mathrm{ch}'_4(A_5) \leqslant 0$.

Let A_6 be a k-face of G, for $6 \leqslant k \leqslant 10$. Note $\mathrm{ch}_4(A_6) = 4 - k$. If there is a 3-vertex x incident with A_6 and a 3-face, then $\mathrm{cd}(x) \leqslant d(n)$. Otherwise, each copy of a vertex in the facial walk of A_6 sends at most $(k-4)/k$ into A_6 by Rules 2, 3, 4, 6, and 7, and therefore $\mathrm{ch}'_4(A_6) \leqslant 0$.

Let A_{11} be an 11-face of G. Note $\mathrm{ch}_4(A_{11}) = -7$. Each vertex sends at most $\frac{2}{3}$ into A_{11}. To prove that $\mathrm{ch}'_4(A_{11}) \leqslant 0$, it suffices to show that some vertex sends at most $\frac{1}{3}$ into A_{11}. If there is a 3-vertex x incident with A_{11} and two $\leqslant 5$-faces, then $\mathrm{cd}(x) \leqslant d(n)$. If A_{11} is incident with eleven 3-vertices, then, since the degree of A_{11} is odd, one of these 3-vertices is not incident with an $\leqslant 5$-face, and sends only $\frac{1}{3}$ into A_{11} by Rule 4. If A_{11} is incident with an $\geqslant 4$-vertex, then that vertex sends at most $\frac{1}{3}$ into A_{11} by Rules 6 and 7. In either case, the desired vertex has been found, and thus, $\mathrm{ch}'_4(A_{11}) \leqslant 0$.

Let A_{12} be a k-face of G, for $12 \leqslant k \leqslant 15$. If there is a 3-vertex x incident with A_{12} and two 3-faces, then $\mathrm{cd}(x) \leqslant d(n)$. Otherwise, each copy of a vertex sends at most $\frac{2}{3}$ into A_{12}, and $\mathrm{ch}'_4(A_{12}) \leqslant 0$.

Let A_{16} be a k-face of G of degree $k \geqslant 16$. In this case, $n \geqslant 16$. If there is a 3-vertex x incident with A_{16} and two 3-faces, then $\mathrm{cd}(x) \leqslant d(n)$. Let C be a cycle disjoint from

G of the same length as the facial walk of A_{16}. Let S_2 be the set of vertices of C which correspond to 2-vertices of G, let C_2 be the subgraph of C induced by S_2, and let m be the number of components of C_2. If $m=0$, then each copy of a vertex in the facial walk of A_{16} sends at most $\frac{2}{3}$ into A_{16}, and $\mathrm{ch}_4'(A_{16})\leqslant 0$.

If $m\geqslant 6$, then let $G_1,\ldots,G_m$ be the components of $C-S_2$. Note that if $|V(G_i)|=1$, then the copy of the vertex x of G corresponding to G_i sends at most $\frac{1}{3}$ into A_{16} by Rule 4. Otherwise, G_i has at least two vertices, each of which sends at most $\frac{2}{3}$ into A_{16}. Regardless, each G_i sends at most $|V(G_i)|-\frac{2}{3}$ into A_{16}. Since $m\geqslant 6$, then A_{16} receives at most $k-4$, and $\mathrm{ch}_4'(A_{16})\leqslant 0$.

If $1\leqslant m\leqslant 5$, then let $J_1,\ldots,J_m$ be the components of C_2, and let $H_1,\ldots,H_m$ be the corresponding subgraphs of G. For $1\leqslant i\leqslant m$, let $j_i=|V(J_i)|$. Let $j:=\sum_{i=1}^m j_i$.

Let an $i\in\{1,\ldots,m\}$ be given. Let x be a vertex of H_i. Thus, $\mathrm{cd}(x)\leqslant k+n-j_i-3$. Since $\mathrm{cd}(x)>d(n)$, it follows that $k-j_i>\frac{4}{5}n+2$, or $k-j_i\geqslant\frac{4}{5}n+\frac{11}{5}$. Summing over i gives $j\leqslant m(k-\frac{4}{5}n-\frac{11}{5})$, and therefore, $k-j\geqslant k-m(k-\frac{4}{5}n-\frac{11}{5})$.

Since $k\leqslant n$ and $m\geqslant 1$, it also follows that $(m-1)(n-k)\geqslant 0$, and thus $k-j\geqslant\frac{11}{5}m+(1-\frac{m}{5})n$. Remembering that $m\leqslant 5$ and $n\geqslant 16$ gives $(1-m/5)n\geqslant(1-m/5)16$. Combining the last two inequalities gives $k-j\geqslant 16-m$. If $k-j\geqslant 12$, then at least 12 vertices incident with A_{16} have degree at least three in G, and each sends at most $\frac{2}{3}$ into A_{16}; thus A_{16} receives at most $k-4$. Otherwise, $m=5$, and $k-j=11$. Note that if one of these eleven vertices sends at most $\frac{1}{3}$, then again, A_{16} receives at most $k-4$. Also, if a vertex x sends either $\frac{2}{5}$ or $\frac{1}{2}$ into A_{16}, one of the neighbors of x has degree at least three and also sends at most $\frac{1}{2}$ into A_{16}, and again, A_{16} receives at most $k-4$. The only case that remains is that each of the vertices of degree at least three sends $\frac{2}{3}$, which is impossible since $k-j$ is odd. In each case, $\mathrm{ch}_4'(A_{16})\leqslant 0$.

Thus every face of G has non-positive modified charge.

These two results combine to give the inequality $\sum_{x\in V\cup F}\mathrm{ch}_4'(x)\leqslant 0$. Since $\sum_{x\in V\cup F}\mathrm{ch}_4(x)=\sum_{x\in V\cup F}\mathrm{ch}_4'(x)$, this contradicts Lemma 2.1. Thus the assumption that G has no vertex x with $\mathrm{cd}(x)\leqslant d(n)$ is false. This concludes the proof that every graph in $D(\Sigma_0,n)\cup D(\tilde{\Sigma}_1,n)$ has a vertex with cyclic degree at most $d(n)$.

To show that the result is best possible, consider the following. Let the embedded graphs $G_3,\ldots,G_7,G_{10}$ be defined, respectively, by the six graphs of Fig. 1, where the embedding is taken to be in the projective plane by identifying antipodal points of the circle. For j satisfying $j\geqslant 8$ and $j\neq 10$, let G_j be derived from G_{10} as follows: Let m and n be the integers satisfying $j=5n+m$, $n\geqslant 1$, and $1\leqslant m\leqslant 5$. Let $E_1,\ldots,E_5$ be the dotted, dashed, bold, bold dotted, and bold dashed edges of G_{10}, respectively (note that each set has three edges). Let G_j be the graph where each edge in $\bigcup_{i=1}^m E_i$ is replaced by a path on $n+1$ vertices, and each edge in $\bigcup_{i=m+1}^5 E_i$ is replaced by a path on n vertices. For each $j\geqslant 3$, the graph G_j is a projective plane graph with maximum face degree j and minimum cyclic degree $d(j)$. Note that G_3 is K_6, and G_5 is the Petersen graph.

For each $j\geqslant 3$, form a plane graph H_j from G_j as follows: Reinterpret each embedding of Fig. 1 as on a disk D. Let S be a sphere of the same radius as D which

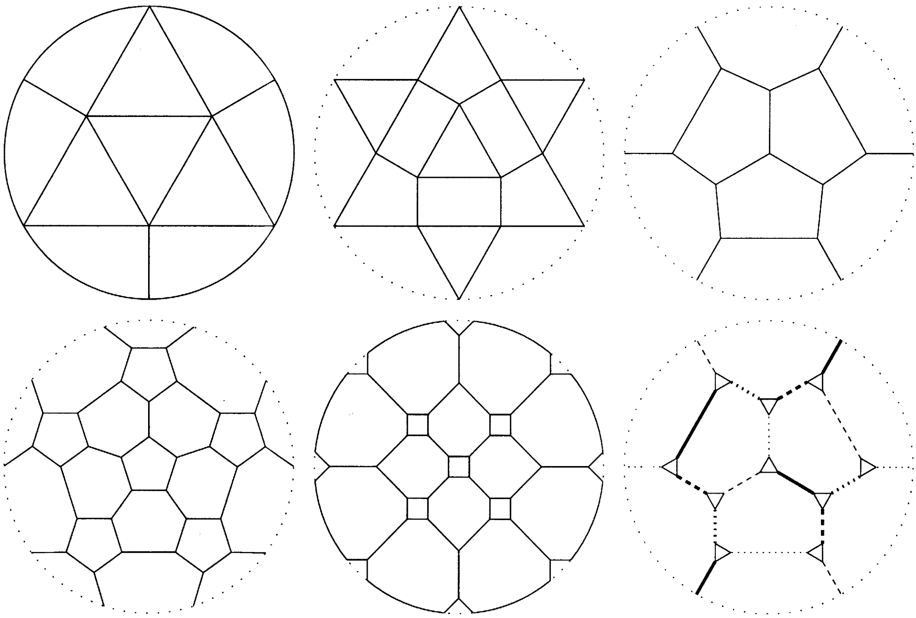

Fig. 1. Some extremal projective plane graphs.

intersects D on its boundary. Project G_j onto the top half of S. Rotate D halfway around in the plane, and project G_j onto the bottom half of S. This graph H_j is a plane graph (or sphere graph) with maximum face degree j and minimum cyclic degree $d(j)$. Note that H_3 is the icosahedron, and H_5 is the dodecahedron. $\square$

The next result is a bound on the cyclic chromatic number of plane and projective plane graphs in terms of their maximum face degree.

Corollary 3.2. *The cyclic chromatic number of* $D(\Sigma_0, \Delta)$ *and of* $D(\tilde{\Sigma}_1, \Delta)$ *is at most* $d(\Delta) + 1$. *For* $\Delta \in \{3, 5\}$, *the cyclic chromatic number of* $D(\tilde{\Sigma}_1, \Delta)$ *equals* $d(\Delta) + 1$.

Proof. If J is a graph, let $\mathrm{cc}(J)$ be the cyclic chromatic number of J.

Let G be a graph embedded on the plane or projective plane on the fewest vertices with $\mathrm{cc}(G) \geqslant d(\Delta(G)) + 2$. Clearly $v(G) \geqslant 4$. By Theorem 3.1, there is a vertex x of G with $\mathrm{cd}(x) \leqslant d(\Delta)$. Let H be G with an edge incident with x contracted. Clearly, H is connected, $\Delta(H) \leqslant \Delta(G)$, and from the minimality of G, $\mathrm{cc}(H) \leqslant d(\Delta(G)) + 1$. This gives a partial cyclic coloring of G, and since $\mathrm{cd}(x) \leqslant d(\Delta)$, it may be colored a color different from its neighbors, a contradiction.

To see that the result is best possible for $\Delta \in \{3, 5\}$ on the projective plane, consider the graphs G_3 and G_5; each has $d(\Delta) + 1$ pairwise cyclically adjacent vertices. $\square$

4. Extensions to diagonal colorings

Hornak and Jendrol [9,10] (in terms of the dual notion of face colorings) introduced an interesting generalization both of cyclic colorings and of diagonal colorings [8]. Let two vertices x, y of an embedded graph G be *d-diagonally adjacent* if there is a set S of edges such that $|S| \leqslant d$, and such that x and y are incident with a common face of $G - S$. Let a *d-diagonal* coloring of a graph G be a coloring of the vertices of G, such that each pair of d-diagonally adjacent vertices receive different colors. Thus a cyclic coloring is a 0-diagonal coloring. Given a set T of embedded graphs, the *d-diagonal chromatic number* of T is the minimum number k, such that every graph in T has a d-diagonal k-coloring.

Results on the 1-diagonal chromatic number of triangulations have been proven by Bouchet et al. [8], by Borodin [3,5], and by Sanders and Zhao [15]. The conjecture that plane triangulations have 1-diagonal chromatic number equal to nine is known as the nine color conjecture in [11] (p. 48). Hornak and Jendrol [9] and Sanders and Zhao [16] proved results respectively on the 1-diagonal and d-diagonal chromatic number of quadrangulations.

There have been some results for the general case. For $\Delta \geqslant 11$, Hornak and Jendrol [10] gave an upper bound of $1 + (2\Delta - 4)(\Delta - 1)^d$ for plane graphs. For $8 \leqslant \Delta \leqslant 10$, Sanders and Zhao [17] obtained the same bound. In [16,17] they showed upper bounds for $\Delta = 3, 4, 5, 6$, and 7 of $2 + \frac{19}{4} \cdot 2^d$, $1 + 8 \cdot 3^d$, $1 + 12 \cdot 4^d$, $1 + \frac{49}{5} \cdot 5^d$, and $1 + 11 \cdot 6^d$, respectively.

It can be shown that a vertex of cyclic degree k has d-diagonal degree (defined analogously) at most $k(\Delta - 1)^d$. Thus Theorem 3.1 implies that a plane or projective plane graph has a vertex of d-diagonal degree at most $d(\Delta)(\Delta - 1)^d$. Unfortunately, this does not yield a coloring result similar to Corollary 3.2, since contracting an edge incident with an $\geqslant 3$-vertex may prevent vertices which were previously d-diagonally adjacent from being so in the new graph.

Careful examination of the proof of Theorem 3.1 does yield a d-diagonal coloring result for $\Delta \geqslant 16$. Let an edge α of an embedded graph G be *deletable* if $\Delta(G - \alpha) = \Delta(G)$. Note that if $\Delta \geqslant 16$, in the proof of Theorem 3.1, each time a vertex x is found with $\mathrm{cd}(x) \leqslant d(\Delta)$, either $\deg(x) \leqslant 2$ or the graph contains a deletable edge. It is a short exercise to show that a graph on the fewest edges with no d-diagonal $1 + k(\Delta - 1)^d$-coloring has no deletable edge and no vertex x with $\deg(x) \leqslant 2$ and $\mathrm{cd}(x) \leqslant k$. Thus plane and projective plane graphs with $\Delta \geqslant 16$ have d-diagonal $1 + (\lfloor \frac{9}{5}\Delta \rfloor - 1)(\Delta - 1)^d$-colorings.

Thus combining this with the previous results yields:

Theorem 4.1. *Every plane or projective plane graph with $\Delta \geqslant 7$ has a d-diagonal $1 + (\lfloor \frac{9}{5}\Delta \rfloor - 1)(\Delta - 1)^d$-coloring.*

Lower bounds for this problem were given by Sanders and Zhao in [16].

The final two sections will only be concerned with plane graphs, as connectivity arguments are used which do not easily generalize to projective planar graphs.

5. Cyclic coloring with $\Delta = 5$

This section will show that plane graphs with $\Delta = 5$ have cyclic 8-colorings. A plane graph G is (Δ,k)-*minimal* if it has $\Delta(G) \leqslant \Delta \leqslant k$, has no cyclic k-coloring, and is a graph with the sum of its vertices and edges minimal with respect to this property.

Lemma 5.1. *For $\Delta \leqslant k$, no (Δ,k)-minimal graph contains a deletable edge.*

Proof. Let integers Δ and k be given. Let G be a (Δ,k)-minimal graph with deletable edge α. By definition, $\Delta(G-\alpha) = \Delta(G) \leqslant \Delta$, and thus $G-\alpha$ has a cyclic k-coloring by the (Δ,k)-minimality of G. Clearly, this is also a cyclic k-coloring of G, contradicting the (Δ,k)-minimality of G. $\square$

Since a bound on the minimum cyclic degree is not enough to obtain this coloring result, a technique sometimes known as reducible configurations will be used. Loosely, a reducible configuration is a structure which is not present in a (Δ,k)-minimal graph, for some values of Δ and k. Thus a deletable edge may be considered a reducible configuration for cyclic coloring. Note, in particular, that a $(5,8)$-minimal graph has no edge incident with a 3-face and an $\leqslant$4-face.

A k-*cut* of a graph G is a set S of k vertices of G such that $G-S$ is not connected.

Let G be a graph embedded on a surface Σ, and let C be a separating closed curve on Σ, such that $(C \cap G) \subset V(G)$, and such that $\Sigma \backslash C$ has two components Σ_1, Σ_2 which each intersect G. Then a C-*patch* is $G \cap (\Sigma_i \cup C)$, for i either 1 or 2.

Lemma 5.2. *For $\Delta \leqslant k$, no (Δ,k)-minimal graph has a 1-cut.*

Proof. Let integers $\Delta \leqslant k$ be given. Let G be a (Δ,k)-minimal graph with a 1-cut S.

Let C be a closed curve in the plane such that $C \cap G = S$, and such that neither of the two C-patches P_1, P_2 has vertex set equal to S. Clearly for $i \in \{1,2\}$, $\Delta(P_i) \leqslant \Delta$ and thus P_i has a cyclic k-coloring, by the (Δ,k)-minimality of G. Let A be the face of G such that $(C-S) \subset A$. The colors of P_2 may be permuted to match P_1 at S and to differ from P_1 otherwise on A, giving a cyclic k-coloring of G. $\square$

Thus a (Δ,k)-minimal graph is 2-connected.

The next two lemmas were proved by Borodin [4].

Lemma 5.3 (Borodin [4]). *For $j \leqslant \Delta \leqslant k$, no (Δ,k)-minimal graph has a separating j-circuit.*

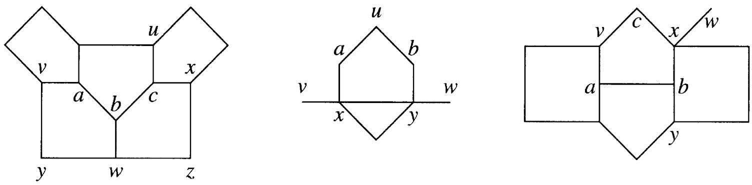

Fig. 2. Three reducible configurations.

Lemma 5.4 (Borodin [4]). *For $\Delta \leqslant k$, no (Δ, k)-minimal graph has a vertex x with* $\mathrm{cd}(x) \leqslant k-1$, *or an edge yz incident with two $\geqslant 4$-faces and such that* $\mathrm{cd}(y) \leqslant k$ *and* $\mathrm{cd}(z) \leqslant k+1$.

A *partial coloring* of a graph G is a coloring of a subset of the vertices of G. A vertex x of a graph G with a partial coloring *sees* a color k if and only if there is a vertex y colored k, such that there is some face A of G incident with both x and y. Let a $(kj_1, \dots, j_k)$-*vertex* be a k-vertex incident with faces $(A_1, \dots, A_k)$ in cyclic order such that $\deg(A_i) = j_i$ for $i \leqslant k$. Given a graph G and an edge α of G, let $G \cdot \alpha$ be G with α contracted (deleting multiple edges thus created).

Lemma 5.5. *No* $(5,8)$-*minimal graph has a* $(3,5,3,5)$-*vertex.*

Proof. Let G be a $(5,8)$-minimal graph containing a $(3,5,3,5)$-vertex x. Let the neighbors of x in cyclic order be (a,b,c,d). Let $H := G \cdot ax \cdot cx$. Clearly H is a plane graph, has $\Delta(H) \leqslant 5$, and thus has a cyclic 8-coloring by the $(5,8)$-minimality of G. This gives an 8-coloring of $G - x$, where a and c receive the same color. But in G, the vertex x now sees at most seven colors; it can be colored with the eighth color, giving a cyclic 8-coloring of G, a contradiction. □

Given a vertex x of a graph G, let $N(x)$ be the set of vertices of G adjacent to x. Let a graph G *contain Configuration* 1 if the following are satisfied. Each vertex in $\{a,b,c\} \subset V(G)$ is a $(5,5,5)$-vertex (and thus has degree three). There is a face bounded by $abwyv$, there is a face bounded by $bcxzw$, and $N(c) = \{b,u,x\}$. Also, either $\mathrm{cd}(w) \leqslant 11$, or both $\mathrm{cd}(w) = 12$, and either $\mathrm{cd}(y) \leqslant 9$, or $\mathrm{cd}(x) \leqslant 10$. (See the first configuration of Fig. 2.)

Let a graph G *contain Configuration* 2 if the following are satisfied. There is a face bounded by $xybua$. The edge xy is incident with a 3-face, $N(x) = \{a,s,v,y\}$, and $\{b,s,w,x\} \subset N(y)$. Also, $\deg(a) = 3$, and b is a $(4,5,5)$-vertex (and thus has degree three). Finally, either a is a $(4,5,5)$-vertex (and thus has degree three), or $\deg(y) = 4$. (See the second configuration of Fig. 2.)

Let a graph G *contain Configuration* 3 if the following are satisfied. There is a face bounded by $abxcv$. There is a face bounded by $ybxst$. Also, $N(x) = \{b,c,s,w\}$, and c is a $(4,5,5)$-vertex (and thus has degree three). (See the third configuration of Fig. 2.)

Lemma 5.6. *No* $(5,8)$*-minimal graph contains Configuration* 1, 2, *or* 3.

Proof. Let G be a $(5,8)$-minimal graph containing Configuration k, for $k \in \{1,2,3\}$. First, notice that from Lemma 5.3, and from the planarity of G, that vertices distance at most five apart in Fig. 2 are actually distinct vertices (e.g. v and x).

If $k=1$, let $G' := G + cz$, and let $H := G' \cdot va \cdot ab \cdot bc \cdot cu \cdot cz$. If $k=2$, and if a is a $(4,5,5)$-vertex, then let $G' := G + av$, and let $H := G' \cdot ua \cdot av$. If $k=2$, and if a is not a $(4,5,5)$-vertex, then let $G' := G + av + bw$, and let $H := G' \cdot ua \cdot av \cdot ub \cdot bw$. If $k=3$, let $G' := G$, and let $H := G' \cdot va \cdot ab \cdot bx \cdot xw \cdot by$.

Clearly H is a plane graph, has $\Delta(H) \leqslant 5$, and thus has a cyclic 8-coloring by the $(5,8)$-minimality of G. This gives an 8-coloring of G', where both ends of each contracted edge of G' receive the same color. If $k=1$, $\mathrm{cd}(w)=12$, and $\mathrm{cd}(y) \leqslant 9$, then let $L := (w,y,a,c,b)$. If $k=1$, $\mathrm{cd}(w)=12$, and $\mathrm{cd}(y) > 9$, then let $L := (w,x,a,c,b)$. If $k=1$, and $\mathrm{cd}(w)$, then let $L := (w,a,c,b)$. If $k=2$, and a is a $(4,5,5)$-vertex, then let $L := (x,b,a)$. If $k=2$, and a is not a $(4,5,5)$-vertex, then let $L := (x,y,a,b)$. If $k=3$, then let $L := (x,a,b,c)$.

Remove the colors from the vertices in L. This yields a coloring of G with some vertices colored, and others not. We need to show that this is a partial cyclic coloring of G, that is, that two distinct vertices colored with the same color are not incident with the same face. This is trivial for most such pairs of vertices, but not so for a pair of colored vertices which were identified in H. For these pairs, since $\Delta(G) \leqslant 5$, Lemma 5.3 gives the result. The uncolored vertices may be colored in the order given in L so that when each is colored, it sees at most seven colors.

For example, if $k=3$, first note that the vertices in $\{v,w,y\}$ are colored with the same color. Then note that $\mathrm{cd}(x) \leqslant 12$, but since the vertices in $\{a,b,c\}$ are uncolored, and since the vertices in $\{v,w,y\}$ are colored the same, then x sees at most seven colors; thus color x a color it does not see. Now note that $\mathrm{cd}(a) \leqslant 9$, the vertices in $\{b,c\}$ are uncolored, and the vertices in $\{v,y\}$ are colored the same, so a now sees at most six colors; thus color a a color it does not see. Next, $\mathrm{cd}(b) \leqslant 9$, c is uncolored, and the vertices in $\{v,y\}$ are colored the same; color b. Finally, $\mathrm{cd}(c) \leqslant 8$, and the vertices in $\{v,w\}$ are colored the same; color c.

This gives a cyclic 8-coloring of G, contradicting its $(5,8)$-minimality. □

When applying Lemma 5.6, to make it clear which configuration is being used, Lemma 5.6.k will be the sublemma that no $(5,8)$-minimal graph contains Configuration k.

Now the discharging method will be applied. Here, it is more convenient to use different initial charges than were used before. Let a 6-charged graph G be *discharged by the second set of rules* if a function ch_6' is defined by modifying ch_6 according to Rules 1–10 below.

1. For each 4-face A_4, and for each vertex x incident with A_4, A_4 receives $\frac{1}{2}$ from x.

Rules 2–10 apply to a 5-face A_5 which has incident vertices in cyclic order (w_1, w_2, w_3, w_4, w_5). Rules 2–9 tell what charge A_5 receives from w_2. To obtain what charge A_5 receives from a vertex $w \in \{w_1, w_3, w_4, w_5\}$, relabel the vertices so that w becomes w_2. Similarly, there are symmetric cases to Rule 10 which are obtained by relabeling as well.

2. If w_2 is a $(4,5,5)$-vertex, then A_5 receives $\frac{5}{4}$ from w_2.
3. If a vertex in $\{w_1, w_3\}$ is an $\geqslant$4-vertex and w_2 is a $(5,5,5)$-vertex adjacent to two $(5,5,5)$-vertices, then A_5 receives $\frac{5}{4}$ from w_2.
4. If each vertex in $\{w_1, w_2, w_3\}$ is a $(5,5,5)$-vertex, then A_5 receives $\frac{1}{2}$ from w_2.
5. If $\deg(w_2) = 3$, and A_5 receives no charge from w_2 by Rules 2, 3, or 4, then A_5 receives 1 from w_2.
6. If $\deg(w_2) = 4$, and an edge in $\{w_1w_2, w_2w_3\}$ is incident with a 3-face, then A_5 receives $\frac{3}{4}$ from w_2.
7. If $\deg(w_2) = 4$, and A_5 receives no charge from w_2 by Rule 6, then A_5 receives $\frac{1}{2}$ from w_2.
8. If $\deg(w_2) = 5$, and each edge in $\{w_1w_2, w_2w_3\}$ is incident with a 3-face, then A_5 receives $\frac{1}{2}$ from w_2.
9. If $\deg(w_2) = 5$, and A_5 receives no charge from w_2 by Rule 8, then A_5 receives $\frac{1}{4}$ from w_2.
10. If each vertex in $\{w_1, w_2, w_3, w_4\}$ is a $(5,5,5)$-vertex, then A_5 receives $\frac{1}{4}$ from the face adjacent to A_5 which is incident with w_2w_3.

The following theorem will show that there is no $(5,8)$-minimal graph, by showing that every plane graph with $\Delta = 5$ has one of the structures present which was shown not to appear in a $(5,8)$-minimal graph in the lemmas above. It follows that each plane graph with $\Delta = 5$ has a cyclic 8-coloring.

Theorem 5.1. *Every plane graph with $\Delta = 5$ has a cyclic 8-coloring.*

Proof. Let G be a $(5,8)$-minimal graph. Let G be 6-charged, and then discharged by the second set of rules. By Lemma 5.4, G has no $\leqslant$2-vertices.

Let x_3 be a 3-vertex of G. Note $\mathrm{ch}_6(x_3) = 3$. By Lemma 5.4, x_3 is either a $(4,5,5)$-vertex or a $(5,5,5)$-vertex. If x_3 is a $(4,5,5)$-vertex, then it sends out $\frac{1}{2}$ by Rule 1 and $\frac{5}{2}$ by Rule 2. If x_3 is a $(5,5,5)$-vertex, then either it sends out $\frac{5}{2}$ by Rule 3 and $\frac{1}{2}$ by Rule 4, or it sends out 3 by Rule 5. In any case, $\mathrm{ch}'_6(x_3) = 0$.

Let x_4 be a 4-vertex of G. Note $\mathrm{ch}_6(x_4) = 2$. By Lemmas 5.1, 5.4, and 5.5, x_4 is incident with at most one 3-face. If x_4 is incident with a 3-face, then it sends out $\frac{3}{2}$ by Rule 6 and $\frac{1}{2}$ by either Rule 1 or Rule 7. If x_4 is not incident with a 3-face, then it sends out 2 by Rules 1 and 7. In either case, $\mathrm{ch}'_6(x_4) = 0$.

Let x_5 be a 5-vertex of G. Note $\mathrm{ch}_6(x_5) = 1$. By Lemma 5.1, if x_5 is incident to at least two 3-faces, it is a $(5,3,5,3,5)$-vertex; here it sends out $\frac{1}{2}$ by each of Rules 8 and 9. Otherwise, x_5 is incident to at least four $\geqslant$4-faces, and it sends out at least 1 by Rules 1 and 9. In either case, $\mathrm{ch}'_6(x_5) \leqslant 0$.

Let x_6 be an $\geqslant$6-vertex of G. Here $\mathrm{ch}_6(x_6)\leqslant 0$, and since no rule sends charge into x_6, $\mathrm{ch}'_6(x_6)\leqslant 0$ as well.

Let A_3 be a 3-face of G. Here $\mathrm{ch}_6(A_3)=0$, and no rules affect A_3, so $\mathrm{ch}'_6(A_3)=0$ as well.

Let A_4 be a 4-face of G. Note $\mathrm{ch}_6(A_4)=-2$, and the only rule that applies is Rule 1, which sends 2 into A_4, and $\mathrm{ch}'_6(A_4)=0$.

Let A_5 be a 5-face of G. Note $\mathrm{ch}_6(A_5)=-4$. Let the vertices incident with A_5 in cyclic order be (w_1,w_2,w_3,w_4,w_5). Let S be the set of all $(5,5,5)$-vertices in $\{w_1,\dots,w_5\}$.

If $S=\{w_1,\dots,w_5\}$, then the only rules that apply are Rules 4 and 10, which send $\frac{5}{2}$ and $\frac{5}{4}$ into A_5, respectively.

Suppose $S=\{w_1,\dots,w_4\}$. Since the faces adjacent to A_5 incident with w_1w_5 and w_4w_5 are 5-faces, then A_5 receives either $\frac{1}{2}$ from w_5 by Rule 7 or $\frac{1}{4}$ from w_5 by Rule 9. Each vertex in $\{w_2,w_3\}$ sends $\frac{1}{2}$ into A_5 by Rule 4. Also, A_5 receives precisely $\frac{1}{4}$ by Rule 10. Each vertex in $\{w_1,w_4\}$ sends at most $\frac{5}{4}$ into A_5. If the sum of the charge received from $\{w_1,w_4,w_5\}$ is more than $\frac{11}{4}$, then Configuration 1 can be seen to be present by mapping a to w_3, b to w_4, w to w_5, etc. By Lemma 5.6.1, this cannot occur, and the sum of the charge received from $\{w_1,w_4,w_5\}$ is at most $\frac{11}{4}$.

Suppose $S=\{w_1,w_2,w_3\}$. By Lemma 5.4, neither vertex in $\{w_4,w_5\}$ is a 3-vertex. Clearly, w_2 sends $\frac{1}{2}$ into A_5 by Rule 4. By Lemma 5.6.1, the sum of the charge received from $\{w_1,w_5\}$ (symmetrically, from $\{w_3,w_4\}$) is at most $\frac{7}{4}$.

Suppose $S=\{w_2,w_4,w_5\}$. Thus A_5 is adjacent only to 5-faces, and neither vertex in $\{w_1,w_3\}$ is a 3-vertex. Clearly w_2 sends 1 into A_5 by Rule 5. By Lemma 5.6.1, the sum of the charge received from $\{w_1,w_5\}$ (symmetrically, from $\{w_3,w_4\}$) is at most $\frac{3}{2}$.

Suppose $S=\{w_4,w_5\}$. By Lemma 5.4, neither vertex in $\{w_1,w_3\}$ is a 3-vertex. The difference of the sum of the charge received from $\{w_4,w_5\}$ and the charge sent to the face adjacent to A_5 incident with w_4w_5 is at most $\frac{9}{4}$.

Assume w_2 is a $(4,5,5)$-vertex. Thus w_2 sends $\frac{5}{4}$ into A_5. By Lemmas 5.4 and 5.6.3, each vertex in $\{w_1,w_3\}$ is an $\geqslant$5-vertex, and thus sends at most $\frac{1}{4}$ into A_5, since in this case, A_5 is not adjacent to a 3-face.

Assume w_2 is not a $(4,5,5)$-vertex. In this case, each vertex in $\{w_1,w_2,w_3\}$ sends at most $\frac{3}{4}$ into A_5. By Lemma 5.5, the sum of the charge received from $\{w_1,w_2,w_3\}$ is at most 2. Note that neither vertex in $\{w_1,w_3\}$ sends charge into A_5 by Rule 8. If neither vertex in $\{w_4,w_5\}$ sends $\frac{5}{4}$ into A_5 by Rule 3, then the sum of the charge received from $\{w_4,w_5\}$ is 2. Thus assume w_4 sends $\frac{5}{4}$ into A_5 by Rule 3. By Lemma 5.6.1, w_3 does not send $\frac{3}{4}$ into A_5, and if w_3 sends 12 into A_5, then w_1 sends at most $\frac{1}{2}$ into A_5.

Suppose $S=\{w_1,w_3\}$. Each vertex in $\{w_1,w_3\}$ sends 1 into A_5, while each vertex in $\{w_4,w_5\}$ sends at most $\frac{3}{4}$ into A_5, and w_2 sends at most $\frac{1}{2}$ into A_5.

Suppose $S=\{w_2\}$. Here w_2 sends 1 into A_5. If A_5 is not incident with a $(4,5,5)$-vertex, then each vertex in $\{w_1,w_3,w_4,w_5\}$ sends at most $\frac{3}{4}$ into A_5. By Lemma 5.4, neither vertex in $\{w_1,w_3\}$ is a $(4,5,5)$-vertex, and at most one vertex in $\{w_4,w_5\}$ is a $(4,5,5)$-vertex. Assume w_5 is a $(4,5,5)$-vertex. Clearly, in this case, w_1 sends at most $\frac{1}{2}$ into A_5. By Lemma 5.6.2, the sum of the charge received from $\{w_3,w_4\}$ is at most $\frac{5}{4}$.

Suppose A_5 is not adjacent to a $(5,5,5)$-vertex. By Lemma 5.4, A_5 is not adjacent to two adjacent $(4,5,5)$-vertices.

Suppose each vertex in $\{w_1,w_3\}$ is a $(4,5,5)$-vertex. Thus each vertex in $\{w_1,w_3\}$ sends $\frac{5}{4}$ into A_5, while w_2 sends at most $\frac{1}{2}$ into A_5. By Lemma 5.4, each of w_4,w_5 sends at most $\frac{1}{2}$ into A_5.

Suppose precisely w_2 is a $(4,5,5)$-vertex. Thus w_2 sends $\frac{5}{4}$ into A_5. Without loss of generality, let w_2w_3 be incident with a 4-face. By Lemma 5.4, w_3 sends at most $\frac{1}{2}$ into A_5. Each vertex in $\{w_1,w_4,w_5\}$ sends at most $\frac{3}{4}$ into A_5.

Suppose A_5 is not adjacent to a $(4,5,5)$-vertex. Here, it receives at most $\frac{3}{4}$ from each vertex in $\{w_1,\ldots,w_5\}$.

In any case, A_5 receives at most 4, and $\mathrm{ch}_6'(A_5)\leqslant 0$.

The above arguments imply that $\sum_{x\in V\cup F}\mathrm{ch}_6'(x)\leqslant 0$. Since $\sum_{x\in V\cup F}\mathrm{ch}_6(x)=\sum_{x\in V\cup F}\mathrm{ch}_6'(x)$, this contradicts Lemma 2.1. □

6. Cyclic coloring with $\Delta = 6$

This section will show the last result of this article, that plane graphs with $\Delta=6$ have cyclic 10-colorings. The following lemma is a direct consequence of Lemma 5.3.

Lemma 6.1. *No* $(6,10)$*-minimal graph has a* 2*-cut.*

A k-cut S is *j-edge* if the induced subgraph on S has at least j edges.

Lemma 6.2. *No* $(6,10)$*-minimal graph has a* 1*-edge* 3*-cut.*

Proof. Let G be a $(6,10)$-minimal graph with a 3-cut $S:=\{x,y,z\}$, such that $xy\in E(G)$. By Lemma 5.3, $\{xz,yz\}\cap E(G)=\emptyset$.

Let C be a closed curve in the plane such that $C\cap G=S$. Let A_{xy} be the face incident with both x and y containing part of C, and let A_{xz} and A_{yz} be defined similarly. Let P_1,P_2 be the two C-patches. Without loss of generality, $xy\in E(P_1)$. Let $H_1:=P_1+xz$, and let $H_2:=P_2+xy+yz$, in each case embedding the new edges in C. Each H_i has $\Delta\leqslant 6$, and thus has a cyclic 10-coloring by the $(6,10)$-minimality of G.

The colors given by those colorings may be permuted to give a coloring of G, thus contradicting its $(6,10)$-minimality, as follows. In each of H_1 and H_2, permute the colors so that x,y,z are, respectively, colored $1,2,3$. Suppose that H_2 has c vertices other than x and z which are incident with A_{xz} in G. Clearly, $1\leqslant c\leqslant 3$. Permute the colors of these vertices to the colors $4,\ldots,3+c$. Next, the vertices other than x and z of H_1 which are incident with A_{xz} in G may be permuted so that they are colored with colors from $\{2,4+c,\ldots,7\}$.

Suppose that H_1 has d vertices other than y and z which are incident with A_{yz} in G. Again, $1\leqslant d\leqslant 3$. The colors of these vertices may be permuted to receive colors from $\{4+c,\ldots,7+d\}$. As $c\leqslant 3$, there is a color $k\in\{4+c,\ldots,7+d\}$ not received by any

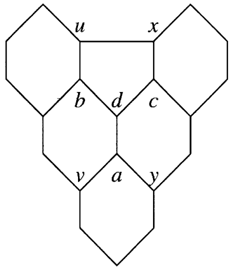

Fig. 3. A reducible configuration.

of these vertices. If $d=3$, let $K=\{1,4,\ldots,3+c,k\}$, else let $K=\{1,4,\ldots,3+c,k,8+d,\ldots,10\}$. In either case, the vertices other than y and z of H_2 which are incident with A_{yz} in G may be permuted so that they are colored with colors from K. The colorings are now as desired. □

Let a graph G *contain Configuration* 4 if there is a face bounded by $bdcxu$, $N(a)=\{d,v,y\}$, and each vertex in $\{b,c,d\}$ is a 3-vertex. See the configuration of Fig. 3 (which shows faces of maximum possible degree).

Lemma 6.3. *No* $(6,10)$*-minimal graph contains Configuration* 4.

Proof. Let G be a $(6,10)$-minimal graph containing Configuration 4. Let $G':=G+cy$, and let $H:=G'\cdot ub\cdot bd\cdot da\cdot av\cdot xc\cdot cy$. Clearly H is a plane graph, and has $\Delta\leqslant 6$, and thus has a cyclic 10-coloring by the $(6,10)$-minimality of G. This gives a 10-coloring of G, where both ends of each contracted edge of G' receive the same color. Remove the colors from the vertices in $\{a,b,c,d\}$. By Lemmas 6.1 and 6.2, u and v are not cyclically adjacent, and x and y are not cyclically adjacent. Thus this coloring is a partial cyclic coloring of G. The uncolored vertices can be colored so that when each is colored, it sees at most nine colors. This gives a cyclic 10-coloring of G, contradicting its $(6,10)$-minimality. □

Now a final application of the discharging method will appear. This section will return to the initial charges which were used in Section 3. Let a 4-charged graph G be *discharged by the third set of rules* if a function ch_4'' is defined by modifying ch_4 according to Rules 1–4 below.

1. Let each 3-vertex send $\frac{1}{3}$ into each of its incident 6-faces.
2. Let each 3-vertex send $\frac{1}{6}$ into each of its incident 5-faces.
3. For each edge from a 3-vertex x to an $\geqslant$4-vertex y, and for each 6-face A incident with xy, let x send $\frac{1}{6}$ through y into A.
4. For each edge α incident with both a triangle T and a 6-face A, and for each $\geqslant$4-vertex x incident with α, let T send $\frac{1}{6}$ through x into A.

Theorem 6.1. *Every plane graph with $\Delta = 6$ has a cyclic 10-coloring.*

Proof. Let G be a $(6,10)$-minimal graph. Let G be 4-charged, and then discharged by the third set of rules. By Lemmas 5.2 and 6.1, G has no $\leqslant 2$-vertices.

Let x_3 be a 3-vertex of G. Note that $\mathrm{ch}_4(x_3) = 1$. By Lemma 5.4, x_3 is not incident with a triangle. Assume x_3 is incident with a 4-face A_1. By Lemma 5.4, x_3 is incident with two 6-faces. Thus x_3 sends out $\frac{2}{3}$ by Rule 1. By Lemma 5.4, the neighbors of x_3 incident with A_1 have degree at least four. Thus x_3 sends out at least $\frac{1}{3}$ by Rule 3.

Assume x_3 is incident with two 5-faces. By Lemma 5.4, x_3 is incident with a 6-face A_2. Thus x_3 sends out $\frac{1}{3}$ by Rule 1 and $\frac{1}{3}$ by Rule 2. By Lemma 5.4, the neighbors of x_3 incident with A_2 have degree at least four. Thus x_3 sends out $\frac{1}{3}$ by Rule 3.

Assume x_3 is incident with one 5-face and two 6-faces. Thus x_3 sends out $\frac{2}{3}$ by Rule 1 and $\frac{1}{6}$ by Rule 2. By Lemma 6.3, x_3 has a neighbor of degree at least four. Thus x_3 sends out at least $\frac{1}{6}$ by Rule 3.

If x_3 is incident with three 6-faces, then it sends out 1 by Rule 1. In any case, x_3 sends out at least 1. Clearly, it does not receive any charge by the rules. Thus $\mathrm{ch}_4''(x_3) \leqslant 0$.

Let x_4 be an $\geqslant 4$-vertex of G. Note that $\mathrm{ch}_4(x_4) \leqslant 0$, and the rules do not send any charge into x_4 (only through x_4). Thus $\mathrm{ch}_4''(x_4) \leqslant 0$ as well.

Let A_3 be a 3-face of G. Note that $\mathrm{ch}_4(A_3) = 1$. By Lemma 5.1, A_3 is not incident with a 3-vertex. By Lemma 5.1, each of the 3 edges incident with A_3 is incident with a 6-face. Thus A_3 sends out 1 by Rule 4, and $\mathrm{ch}_4''(A_3) = 0$.

Let A_4 be a 4-face of G. Note that $\mathrm{ch}_4(A_4) = 0$, and the rules do not send any charge into A_4. Thus $\mathrm{ch}_4''(A_4) = 0$ as well.

Let A_5 be a 5-face of G. In this case, $\mathrm{ch}_4(A_5) = -1$. Clearly A_5 receives at most $\frac{5}{6}$ by Rule 2, and none from the other rules. This gives that $\mathrm{ch}_4''(A_5) \leqslant -\frac{1}{6}$.

Let A_6 be a 6-face of G. Here, $\mathrm{ch}_4(A_6) = -2$. Each vertex incident with A_6 either sends $\frac{1}{3}$ into A_6 by Rule 1 and none from the others or sends at most $\frac{1}{3}$ into A_6 by Rules 3 and 4 and none from the others. Thus A_6 receives at most 2 from these vertices, and $\mathrm{ch}_4''(A_6) \leqslant 0$.

The above arguments imply that $\sum_{x \in V \cup F} \mathrm{ch}_4''(x) \leqslant 0$. Since $\sum_{x \in V \cup F} \mathrm{ch}_4(x) = \sum_{x \in V \cup F} \mathrm{ch}_4''(x)$, this contradicts Lemma 2.1. □

Given a surface Σ, let $\mathrm{cc}(\Sigma, n)$ be the cyclic chromatic number of $D(\Sigma, n)$. Appel and Haken [1] (see also [14]) proved that $\mathrm{cc}(\Sigma_0, 3) = 4$. The remainder of the results in the lemma below were proven by Borodin [2,4] (see also [6]).

Lemma 6.4 (Appel and Haken [1] and Borodin [2,4]). $\mathrm{cc}(\Sigma_0, 3) = 4$, $\mathrm{cc}\ (\Sigma_0, 4) = 6$, $\mathrm{cc}(\Sigma_0, 7) \leqslant 12$, *and* $\mathrm{cc}(\Sigma_0, k) \leqslant 2k - 3$ *for* $k \geqslant 8$.

The following corollary sums up the best known results on the cyclic chromatic number of plane graphs.

Corollary 6.2. *For all* Δ, $\lfloor \frac{3}{2}\Delta \rfloor \leqslant \mathrm{cc}(\Sigma_0, \Delta) \leqslant \lfloor \frac{9}{5}\Delta \rfloor$, *and if* $\Delta \in \{3, 4, 5, 8, 9, 10\}$, *then* $\mathrm{cc}(\Sigma_0, \Delta) \leqslant \lfloor \frac{9}{5}\Delta \rfloor - 1$.

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